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On the scattering of aerodynamic noise

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According to the Lighthill acoustic analogy, the sound induced by a region of turbulence is the same as that due to an equivalent distribution of quadrupole sources within the fluid. It is known that the presence of scattering bodies situated near such multipoles can convert some of their intense near field energy into the form of sound waves whose amplitude is far greater than that of the incident field. Calculations are here presented to determine the extent of this conversion, for hard and soft bodies of various shapes, making use of the reciprocal theorem to recast the problem into one of finding the field, near the obstacle, induced by an incident plane wave. If the obstacle is small compared with a wavelength, then its presence is equivalent to an additional dipole (or source) whose greater efficiency as a sound radiator implies that the familiar intensity law $I \propto U^8$, for far field intensity I against typical turbulence velocity U for an unbounded flow, is replaced by $I \propto U^6$ (or $I \propto U^4$) for a hard (or soft) body. For the situation where the scatterer is large compared with wavelength, the prototype problem of a wedge of exterior angle $(p/q)\pi$ is shown to yield an intensity law $I \propto U^{4+2q/p}$ for both hard and soft surfaces. This result is shown to hold for the more general 'wedge-like' surfaces, whose dimensions are large scale and whose edges may be smoothed out on a small scale, compared with wavelength. The method used involves the matching of an incompressible flow, on the fine scales typical of the edge geometry, to an outer flow determined by the large scale features of the surface. Favourable comparisons are made with previous results pertaining to the two-dimensional semi-infinite duct and to the half-plate of finite thickness.

1. Introduction

According to the Lighthill (1952) theory of aerodynamic noise, a region of turbulence in an otherwise quiescent medium is acoustically equivalent to a distribution of quadrupole sources of strength T_{ij}, proportional to the Reynolds stress terms $\rho u_i u_j$, wherein ρ denotes density and u_i the turbulence velocity. The *quadrupole* nature of this equivalent source distribution has been shown by Lighthill to be of great significance, since such a multipole distribution is relatively inefficient, as a means of propagating sound waves to large distances, compared with dipole and even more efficient monopole sources. This property is due essentially to the tendency of cancellation between the constituent monopoles that form a multipole source, and has led to the formulation of Lighthill's celebrated 'U^8 law' giving the functional form of the intensity

$I \propto U^8$ of sound propagating towards infinity, in terms of the typical velocity fluctuation U of turbulence in an unbounded medium. Since U is typically very small, compared with wave speed c, the high index eight that occurs in the intensity law reveals the inefficiency of turbulence as a producer of sound.

It is now well known that the presence of a scattering obstacle can cause a considerable increase in the intensity of the noise field. For such a body is acoustically equivalent to a surface layer of monopoles and dipoles, these being potentially more efficient producers of far field noise than the incident quadrupole source distribution. The extent to which this increase of sound energy is effected depends crucially upon the boundary conditions associated with the scattering body, and upon its shape. Of the results that have been obtained to date, those due to Curle (1955), Powell (1960), and Ffowcs Williams & Hall (1970) are relevant to the present note and are now described briefly.

The effect of surfaces on aerodynamic noise was first discussed by Curle (1955) who showed that the presence of a 'hard' (i.e. rigid) body is equivalent to a layer of *dipoles* over its surface. In particular, if the obstacle is finite, with dimensions small compared with a typical wavelength, then the surface dipoles are effectively in phase and add together to produce an equivalent single dipole whose strength is proportional to the total force on the body. This analysis predicts a sound field larger than that due to the incident quadrupoles alone, with an intensity law $I \propto U^6$ in place of Lighthill's eighth power law; a similar argument for the 'soft' surface, on which the pressure fluctuation is zero, shows the presence of the obstacle to be equivalent to a *monopole* source, with intensity $I \propto U^4$.

Powell (1960) has pointed out that there is no such enhancement of the sound if the obstacle is a hard (or soft) infinite plane. For such a surface is obviously equivalent to an 'image' distribution of quadrupoles of equal (or opposite) strength to the incident quadrupoles, and the U^8 law holds for the scattered field.

The results of Ffowcs Williams & Hall (1970), on the other hand, show that a semi-infinite plate of zero thickness enhances the sound field of quadrupoles near its edge to an extent far greater than that predicted by the general theory of Curle. Specifically, an intensity law $I \propto U^5$ is established for either the hard or soft boundary condition.

Evidently the general arguments based on equivalent monopole and dipole source layers can be misleading without more specific details regarding their strength, if the scattering surface is large compared with wavelength; for such arguments lead to an overestimate for the scattered field in the case of the infinite plane or soft semi-infinite plane, and an underestimate for the hard, semi-infinite plane. The aim of the present work is to predict the form of the far field scattered by hard, or soft, obstacles of various shapes. The problem is posed as one of diffraction theory, with an incident quadrupole field of strength assumed known, and it is required to calculate the scattered field at large distance from the disturbance.

It is worthwhile to emphasize exactly the questions that this kind of work aims to answer. We must stress that it is quite impossible to predict in general what would happen if an obstacle were suddenly placed in a laminar or turbulent flow, even to the extent of establishing an order of magnitude for the increase of sound power. One important question, however, can quite definitely be answered. The fundamental Lighthill theory deals with a region of turbulence in an unbounded medium and predicts the sound power in terms of the stress tensor T_{ij}. The present work assumes similarly that we can measure T_{ij} for the flow in the presence of certain given boundaries, and is concerned with the effect of such surfaces upon the sound field produced by the quadrupole distribution of density proportional to T_{ij}. In particular, it is shown that it is quite inadequate in most cases to calculate merely the 'incident field', by evaluating the Lighthill integral over the source region.

The problem is simplified by assuming time periodic motions throughout, with angular frequency ω; the general time dependence can, in principle, be generated by superposition of periodic solutions according to the usual Fourier transformation. Working with the velocity potential, the periodic nature is implied by writing the potential in the form $\Re[\phi(\mathbf{x})\exp(-i\omega t)]$ and the notation ϕ_i and ψ is used to denote the incident and scattered potentials, with total field given by $\phi = \phi_i + \psi$. The wave equation for ψ reduces to the Helmholtz equation

$$(\nabla^2 + k^2)\,\psi(\mathbf{x}) = 0, \tag{1.1}$$

where $k = \omega/c$ is the acoustic wave-number, c the wave speed; equation (1.1) has to be solved subject to the appropriate boundary condition on the scattering surface S, and an outgoing wave condition at infinity. The wave motion is forced by an incident field corresponding to a distribution of quadrupole sources, which can be generated in terms of simple monopole sources as follows.

An incident monopole point source situated at the point $y_j\,(j = 1, 2, 3)$ has the potential

$$\phi_i^{(\text{source})} = m\,e^{ikR}/R, \tag{1.2}$$

where m is a constant and $R = |x_j - y_j|$ is the distance from source at y_j to observer at x_j. A dipole of strength, or moment, $d_j = ml_j$ is formed by placing a source of strength $-m$ at y_j together with a source of strength $+m$ at a neighbouring point $y_j + l_j$, where the separation $l = |l_j|$ is small compared with wavelength. Thus in the limit of small kl, the potential is given by direct differentiation as

$$\phi_i^{(\text{dipole})} = d_j \frac{\partial}{\partial y_j}\left(\frac{e^{ikR}}{R}\right) = -d\cos\theta\left(\frac{ik}{R} - \frac{1}{R^2}\right)e^{ikR}, \tag{1.3}$$

where the double suffix summation convention is understood, $d = |d_j| = ml$ and θ is the angle between the vectors d_j and $(x_j - y_j)$. Similarly, an incident quadrupole of strength Q_{jk} has potential of the form

$$\phi_i^{(\text{quad})} = Q_{jk}\,(\partial^2/\partial y_j\,\partial y_k)\,(e^{ikR}/R). \tag{1.4}$$

At large values of $R = |x_j - y_j|$, the expression (1.3) takes the form

$$\phi_i^{(\text{dipole})} \sim -dik\cos\theta\,e^{ikR}/R \quad (kR \gg 1), \tag{1.5}$$

and this 'far-field' approximation gives a measure of the energy radiated towards infinity in the form of sound waves. It is seen from (1.2) and (1.5) that the ratio of far fields induced by source and dipole is given by

$$|\phi^{(\text{dipole})}/\phi^{(\text{source})}| \sim kl\cos\theta \quad (kR \gg 1,\ kl \ll 1), \tag{1.6}$$

which is small since kl is small; thus the source (of order unity with respect to wave-number) is more efficient than the dipole (of order kl), this being by a similar argument more efficient than a quadrupole (of order k^2l^2), and so on.

With the aid of these preliminary results, the general effect of small scattering bodies upon an incident quadrupole sound field is easily seen. An elementary application of Green's theorem leads to the identity, the 'Helmholtz formula', that

$$\phi(\mathbf{x}) = \phi_i(\mathbf{x}) + \frac{1}{4\pi}\int_S \left\{\phi(\mathbf{x}')\frac{\partial}{\partial n'}\left(\frac{e^{ikR}}{R}\right) - \frac{e^{ikR}}{R}\frac{\partial\phi}{\partial n'}(\mathbf{x}')\right\} d\mathbf{x}', \tag{1.7}$$

evaluated over the scattering surface S, where $\mathbf{x}$ is any point outside S, n' is the normal into the fluid and $R = |\mathbf{x}-\mathbf{x}'|$. The integral term, which represents the scattered potential $\psi(\mathbf{x})$ due to the presence of the surface, is seen to be equivalent to a layer of dipoles of density proportional to $\phi(\mathbf{x}')$ and a layer of sources of strength proportional to $\partial\phi/\partial n'$. In particular, if the body is hard, then $\partial\phi/\partial n'$ vanishes to leave only a distribution of dipoles.

Expanding the integral of (1.7) for large values of $|\mathbf{x}| = r$, in order to ascertain the far-field amplitude, we see that for the hard boundary condition, for example,

$$\begin{aligned}\psi(\mathbf{x}) &\equiv -\frac{1}{4\pi}\int_S \phi(\mathbf{x}')\, n_j'\frac{\partial R}{\partial x_j}\left(\frac{ik}{R} - \frac{1}{R^2}\right) e^{ikR}\, d\mathbf{x}' \\ &\sim -\frac{ik}{4\pi}\frac{n}{r^2}\cdot\int_S \mathbf{n}'\,\phi(\mathbf{x}')\, e^{ikR}\, d\mathbf{x}' \quad \text{as} \quad r = |\mathbf{x}| \to \infty,\end{aligned} \tag{1.8}$$

and provided the maximum diameter $2a$ of S is much less than a wavelength, we have

$$\psi(\mathbf{x}) \sim -\frac{ik}{4\pi}\frac{e^{ikr}}{r}\frac{\mathbf{x}}{r}\cdot\int_S \mathbf{n}'\,\phi_0(\mathbf{x}')\, d\mathbf{x}' \quad (kr \gg 1,\ ka \ll 1), \tag{1.9}$$

where $\phi_0(\mathbf{x})$ denotes the limit of ϕ as $k \to 0$, and corresponds to an incompressible flow problem near the rigid surface S.

The explicit determination of the low wave-number potential $\phi \sim \phi_0$ follows a procedure familiar in diffraction theory (see Morse & Feshbach 1953): to obtain ϕ_0 and higher approximations if required, one formally expands ϕ and ϕ_i as series of the form

$$\phi = \phi_0 + ik\,\phi_1 + (ik)^2\,\phi_2/2! + \dots \quad (k|\mathbf{x}| \ll 1),$$

substitutes into the integral identity (1.7) and equates powers of k. For the hard scatterer, one thus obtains

$$\phi_0(\mathbf{x}) = \phi_{i0}(\mathbf{x}) + \left(\frac{1}{4\pi}\right)\int_S \phi_0(\mathbf{x}')\,(\partial/\partial n')\,(1/R)\, d\mathbf{x}',$$

which represents a simple incompressible flow problem. A similar analysis is available also for the soft surface.

Formula (1.9) shows that the scattered potential at large distance is like that due to a dipole whose strength is proportional to the total force on S, this being calculated as though the flow field in the vicinity of S were incompressible.

The dipole nature of the scattered field holds for any incident field and the large effect of rigid bodies upon aerodynamic noise is clear. For if the incident field ϕ_i of formula (1.7) is quadrupole in nature, then it is greatly exceeded by the more efficient dipole radiation generated by the scatterer, this being essentially the result due to Curle (1955) in a slightly different context. Evidently the presence of a foreign body transfers some of the intense near-field energy associated with higher-order multipoles into the form of sound waves.

To complete the estimate (1.9) it is required to calculate the function ϕ_0 in detail; apart from the desirability of obtaining an explicit expression for the amplitude of the far field, it is important to verify that the integral of (1.9) is non-zero, in order that the estimate (1.9) is a sensible one.

It is shown in §2 how the far-field limit, $|\mathbf{x}| \to \infty$, can be taken at a very early stage in the calculations, by appealing to the *reciprocal theorem*, which states that monopole source point $\mathbf{y}$ and observation point $\mathbf{x}$ may be interchanged. Thus one may imagine the source to be at $\mathbf{x}$ with observer at $\mathbf{y}$, whence in the limit $|\mathbf{x}| \to \infty$ the problem reduces to that of finding the potential at $\mathbf{y}$ due to a *plane wave* propagating from the direction of $\mathbf{x}$. In the case of small $k|\mathbf{y}|$ of interest here, the Helmholtz equation, with respect to $\mathbf{y}$, reduces to the Laplace equation and it remains to solve an appropriate incompressible flow problem. A simple illustrative example is given for the case of scattering by a soft, or hard, sphere.

The present approach is well suited for the more difficult case of scattering by bodies that are large compared with wavelength. It is clear that the step from formula (1.8) to (1.9) is no longer valid in this circumstance, and the simple general theory is invalid. A prototype problem of this type has been treated by Ffowcs Williams & Hall (1970), who solve the problem of a quadrupole distribution near the sharp edge of a hard, or soft, semi-infinite plate of zero thickness. One might anticipate that such a sharp edge will provide an efficient mechanism for transforming the near-field energy of quadrupoles into the form of sound waves and this intuitive idea is well established by their analysis. The known Green's function of the problem is utilized and leads to the result that quadrupoles close to the edge, with axes perpendicular to the edge, are greatly enhanced by the presence of the half-plate: if r_0 denotes the distance of a quadrupole from the edge, then the ratio of scattered to incident potentials, at large distance, is of the order $(kr_0)^{-\frac{3}{2}}$, and the increase in sound propagated is far in excess of that predicted by the general theory. The effect of a more general (impedance) boundary condition on the half-plate has recently been discussed by Crighton & Leppington (1970).

Using the reciprocal theorem in the manner outlined above, a simple argument is presented in §3 to obtain the essentials of the results of Ffowcs Williams & Hall for the half plane, and for a wedge of more general exterior angle $p\pi/q$, $1 < p/q \leqslant 2$, for which the potential is increased by the factor $(kr_0)^{-2+q/p}$. In view of these results for scattering by sharp edges it is natural to ask whether the large increase in sound is due entirely to the singularity in curvature at the edge, or to the large extent of the scattering surface, the conclusion of the present work being that the latter feature is an essential property.

The effect of finiteness is considered with special reference to the circular disk, since this seems the simplest geometry that involves a finite sharp edge; it is proposed that the edge induces an increase of order $(kr_0)^{-\frac{3}{2}}$ only if the disk is large compared with wavelength, whereas the general theory of Curle is appropriate if the disk is smaller than a wavelength.

Finally it is conjectured that the results for a wedge remain essentially unchanged if the sharp end is rounded off, provided this smoothing takes place on a scale small compared with wavelength. In support of this hypothesis, favourable comparisons are made with two relevant problems for which exact results are available.

Moreover, a simple closed form expression is indicated for one problem where Jones's (1953) extension of the Wiener–Hopf method provides a solution only through the numerical solution of an infinite system of linear equations with complicated coefficients. The present work provides a solution for these equations, without recourse to numerical computations, and may also be capable of extensions to deal with situations where no current modification of the Wiener–Hopf technique is a possible method of attack.

2. Reciprocal theorem and scattering by small bodies

An incident source, with potential ϕ_i given by (1.2), (1.3) or (1.4), is situated at a point $\mathbf{y}$ near a scattering surface S, and it is required to find the scattered field $\psi(\mathbf{x}; \mathbf{y})$ at great distance from the disturbance. An effective method for taking the far-field limit, $|\mathbf{x}| = r \to \infty$, at an early stage in the calculations is suggested with reference to the *reciprocal theorem* which states that under fairly general boundary conditions, including those for both soft and hard bodies considered in the present work, the potential at $\mathbf{x}$ due to a monopole source at $\mathbf{y}$ is precisely the same as the potential at $\mathbf{y}$ due to a source at $\mathbf{x}$. For higher-order multipoles at $\mathbf{y}$ with observer at $\mathbf{x}$, the problem is therefore equivalent to that of a monopole at $\mathbf{x}$ with an observer measuring the appropriate derivatives at $\mathbf{y}$.

In the limit $|\mathbf{x}| = r \to \infty$, which determines the far field, it is seen that the incident field of the problem reduces to a plane wave of suitable amplitude, propagating from the direction of $\mathbf{x}$. To be specific, if a source of incident potential $\exp\{ik|\mathbf{x}-\mathbf{y}|\}/|\mathbf{x}-\mathbf{y}|$ is situated at $\mathbf{x}$, then as $|\mathbf{x}| = r \to \infty$, with $\mathbf{y}$ fixed, we have

$$\phi_i \sim (e^{ikr}/r)\, e^{ik\boldsymbol{\alpha}.\mathbf{y}},$$

where $\boldsymbol{\alpha} = -\mathbf{x}/r$ is the unit vector in the direction from $\mathbf{x}$ to the origin, and this expression represents a plane wave of amplitude (e^{ikr}/r) in the direction of $\boldsymbol{\alpha}$. Thus we are led to consider the problem of finding the scattered potential $\psi(\mathbf{y})$ at $\mathbf{y}$ induced by an incident plane wave of potential

$$\phi_i = A\, e^{ik\boldsymbol{\alpha}.\mathbf{y}}, \tag{2.1}$$

wherein the amplitude $A = e^{ikr}/r$ contains the dependence on $\mathbf{x}$ and is considered a fixed parameter.

The function ψ so determined gives the far field at $\mathbf{x}$ induced by a monopole at $\mathbf{y}$; the corresponding far field due to multipole sources at $\mathbf{y}$ is then obtained by simply differentiating with respect to the point $\mathbf{y}$. Although this procedure does not yield solutions that could not be obtained by other means, it has the distinct advantage that the 'far-field' limit $|\mathbf{x}| \to \infty$ is taken at the outset, and the potential corresponding to an incident plane wave is, furthermore, much easier to handle than that of an incident source.

Our attention in the work that follows is concerned with sound scattered by obstacles that are placed in the 'near field' of multipole sources, that is, within a fraction of a wavelength, in order to calculate the extent to which the intense near-field energy of such sources is converted into the form of sound waves. The present formulation is ideally suited for this purpose, since in terms of the reciprocal problem we have only to calculate the potential field very close to the body ($k|\mathbf{y}| \ll 1$) due to an incident plane wave of potential given by (2.1). Under the 'near-field' approximation $k|\mathbf{y}| \ll 1$ it is natural to expect that the governing Helmholtz equation (1.1) reduces to the Laplace equation, in a first approximation, and the potential at the points $\mathbf{y}$ of interest close to the obstacle is therefore the solution of an appropriate incompressible flow problem.

These two simple ideas, concerning reciprocity and the incompressible nature of the solution at points $\mathbf{y}$ close to the scatterer, provide the basis for all that follows.

The analysis is straightforward for bodies of dimension small compared with wavelength, and explicit details are provided below for the illustrative example of a sphere of radius a, with $ka \ll 1$. If the scatterer is large, on the other hand, then the task of specifying the incompressible flow problem appropriate to the near-field region close to the body is not quite so simple and this question is deferred until a later section.

Turning to the problem of scattering by a small sphere, the body is taken to be acoustically soft, whence $\phi = 0$ on its surface. If a monopole source is situated at $\mathbf{y}$, then the scattered potential ψ has a far field that is shown above to be determined in terms of the potential $\psi(\mathbf{y})$ scattered by the plane wave given by (2.1).

Under the approximation $kr_0 = k|\mathbf{y}| \ll 1$ the governing Helmholtz equation (1.1) reduces to the Laplace equation for ψ_0 and the boundary condition of zero total potential on the sphere requires $\psi = -A\, e^{ik\boldsymbol{\alpha}\cdot\mathbf{y}} \sim -A$, whence

$$\psi_0 = -A \text{ on the surface } r_0 = a. \tag{2.2}$$

Finally it is required that

$$\psi_0 \to 0 \quad \text{as} \quad r_0 \to \infty, \tag{2.3}$$

and the specifications for ψ_0 are complete. It is perhaps not entirely obvious that the condition (2.3) at infinity is appropriate since the approximation $\psi \sim \psi_0$ is valid only in the near field, $kr_0 \ll 1$, but in the context of the more general expansion scheme $\psi \sim \psi_0 + ik\psi_1 + \dots$ outlined in the introduction the condition (2.3) is obtained in a systematic manner.

The solution for ψ_0 is elementary and is given by

$$\psi_0 = -Aa/r_0. \tag{2.4}$$

It follows that a source of potential $\phi_i^{(\text{source})} = e^{ik|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}|$ situated at $\mathbf{y}$ induces a scattered field whose far field has the asymptotic form

$$\psi^{(\text{source})} \sim -\frac{e^{ikr}}{r}\frac{a}{r_0} \quad (kr \gg 1,\, kr_0 \ll 1), \tag{2.5}$$

where $r = |\mathbf{x}|$ and $r_0 = |\mathbf{y}|$. By direct differentiation with respect to the source point $\mathbf{y}$ it is seen that a radial quadrupole of incident potential

$$\phi_i^{(\text{quad})} = \frac{\partial^2}{\partial r_0^2}\frac{\exp(ik|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|} \tag{2.6}$$

induces a scattered field such that

$$\psi^{(\text{quad})} \sim -\frac{2a}{r_0^3}\frac{e^{ikr}}{r} \quad (kr \gg 1,\, kr_0 \ll 1), \tag{2.7}$$

which result can be verified by means of an exact solution for the problem in terms of spherical Bessel functions. The estimate (2.7) reveals, in particular, the interesting result that the ratio of scattered field to incident field is of the order

$$|\psi/\phi_i| = O(a/r_0^3 k^2) \quad (kr_0 \ll 1), \tag{2.8}$$

which is very large when a and r_0 are held fixed and the wave-number k is small. Quadrupoles other than the radial one ((2.6), (2.7)) will have scattered field negligible compared with (2.7), since differentiation of (2.5) in directions other than the radial one gives zero, to this order of approximation.

A similar analysis for the hard sphere shows a scattered field

$$\psi^{(\text{source})} \sim \frac{e^{ikr}}{r}\left(-\frac{ika^3}{2r_0^2}\cos\theta\right) \quad (kr \gg 1,\, kr_0 \ll 1), \tag{2.9}$$

in place of (2.5), where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$. The corresponding quadrupole field follows by differentiation, whence in particular a radial quadrupole (2.6) induces a field such that

$$\psi^{(\text{quad})} \sim -\frac{3a^3}{r_0^4}\, ik\cos\theta\,\frac{e^{ikr}}{r} \tag{2.10}$$

and

$$|\psi/\phi_i| = O(a^3/r_0^4 k); \tag{2.11}$$

again the sound field is greatly enhanced by the presence of the sphere. In this case, quadrupoles with axes parallel to the surface have a scattered field of strength comparable with the radial quadrupole.

The ideas leading to (2.7)–(2.11) are valid for a body of arbitrary shape, provided its dimension is small compared with wavelength, and the important k dependence in formulae (2.8), (2.11) remains unchanged; the geometrical variables a/r_0^3 and a^3/r_0^4 are simply replaced by terms that depend upon details of the geometry. Such bodies may even have sharp edges, provided suitable edge conditions are applied in order to limit the nature of singularities in such regions, since the Helmholtz formula (1.7), basic to the low wave-number expansions, remains valid. Furthermore, similar arguments may be used to provide explicit details regarding scattering by two-dimensional obstacles such as a circular cylinder.

These results are essentially those presented by Curle (1955) in a slightly different context. For the hard boundary, for example, the increase of potential by the large factor (2.11) proportional to $1/k$ reflects the fact that the presence of a rigid body is acoustically equivalent to a layer of *dipole* sources whose strength is proportional to k and therefore greater than the incident quadrupole source of strength $O(k^2)$; the soft body, on the other hand, is equivalent to a layer of *monopole* sources of strength $O(1)$ compared with the incident quadrupole of strength $O(k^2)$.

3. Surfaces with a sharp edge

Some general conclusions regarding the scattering by bodies of dimensions large compared with wavelength can be inferred without difficulty. If the obstacle is *smooth*, for example, in the sense that its minimum radius of curvature is large on a wavelength scale, then the potential at $\mathbf{y}$, induced by a plane wave propagating from the direction of $\mathbf{x}$, can readily be estimated on the basis of ray theory. In particular, if the source point $\mathbf{y}$ is close to the obstacle (compared with local radius) then the far field is effectively zero at points $\mathbf{x}$ on the 'shadow' side of the obstacle, and is represented by an image field for points $\mathbf{x}$ on the 'illuminated' side, where the sign of the reflected potential carries a + or − sign according as the surface is hard or soft; in this region, the field behaves as if the body were an infinite plane. This is in line with the conclusion reached by Meecham (1965), although the limits of applicability were not clear in that work. The transition region, in the vicinity of the shadow boundary, can be dealt with by appealing to results that are available in the literature on the potential at points $\mathbf{y}$ near the boundary of the shadow cast by an incident plane wave.

It remains to deal with the case where the body has a 'sharp edge' (where the radius is small compared with wavelength) with a nearby source point $\mathbf{y}$, and the rest of this work concerns such configurations.

As a simple prototype problem of this class, the scattering surface is taken to be a rigid semi-infinite plane specified by $x_1 < 0$, $x_2 = 0$. Suppose a simple source of potential $e^{ik|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}|$ is situated at the point

$$\mathbf{y}(y_1 = r_0 \cos\theta_0,\; y_2 = r_0 \sin\theta_0)$$

with an observer at $\mathbf{x}(x_1 = r\cos\theta,\; x_2 = r\sin\theta)$, in the same plane $x_3 = y_3$ perpendicular to the edge, measuring the total potential ϕ. According to the reciprocal theorem discussed earlier the limiting form of the far field, as

$$r = |\mathbf{x}| \to \infty,$$

is obtained in terms of the potential $\phi(\mathbf{y}) = \phi_i(\mathbf{y}) + \psi(\mathbf{y})$ at $\mathbf{y}$ due to an incident plane wave

$$\phi_i = A\exp\{-ik(y_1\cos\theta + y_2\sin\theta)\}, \tag{3.1}$$

wherein $A = e^{ikr}/r$ is regarded as a fixed parameter and appears throughout simply as a constant of proportionality.

Such a wave problem, the 'Sommerfeld problem', is a classical one whose exact solution is well-known (cf. Noble 1958). It is now shown that a great deal of information can be inferred, by appealing to simple incompressible flow arguments, without recourse to an exact analysis, this approach being useful for more complicated situations for which an exact solution is *not* available.

In the vicinity of an edge the potential ϕ is continuous whilst its gradient $\nabla\phi$ becomes infinite. Thus for a fixed value of the wave-number k, the wave equation for the scattered potential $\psi(\mathbf{y})$ takes the approximate form

$$\nabla^2\psi(\mathbf{y}) \approx 0 \quad (kr_0 \ll 1), \tag{3.2}$$

and the potential resembles that of an incompressible flow. So far the argument is like that presented in an earlier section for finite bodies, but the great extent of the semi-infinite plate under discussion here precludes the possibility of assigning a simple boundary condition at infinity for the equivalent incompressible flow problem appropriate to the near field $kr_0 \ll 1$.

The conformal transformation $\zeta = (y_1+iy_2)^{\frac{1}{2}}$, $-\pi < \arg(y_1+iy_2) < \pi$, maps the flow region into the right half ζ plane whence it is readily concluded that the function satisfying the Laplace equation is of the order $r_0^{-\frac{1}{2}}$ as $r_0 \to 0$, independently of precise details of the incident field; more specifically, complex variable theory reveals the fact that the derivates of $\psi(\mathbf{y})$ are of the form

$$\frac{\partial\psi}{\partial y_1} - i\frac{\partial\psi}{\partial y_2} \sim A\frac{B(k)}{(y_1+iy_2)^{\frac{1}{2}}} = A\frac{B(k)}{r_0^{\frac{1}{2}}e^{\frac{1}{2}i\theta_0}} \quad (kr_0 \to 0), \tag{3.3}$$

where A is the proportionality constant of (3.1) and $B(k)$ is an unknown complex constant that depends on the parameter k and on the angle of incidence θ.

For a *rigid* semi-infinite plate, the normal derivative of the total potential is zero on the surface, whence $\partial\psi/\partial y_2$ is certainly non-singular as $r_0 \to 0$ for $\theta_0 = \pm\pi$. Consistency with the general result (3.3) therefore requires that the phase of $B(k)$ must be such that $\partial\psi/\partial y_1$ and $\partial\psi/\partial y_2$ take the forms

$$\partial\psi/\partial y_1 \sim AC(k)\,r_0^{-\frac{1}{2}}\sin(\tfrac{1}{2}\theta_0), \quad \partial\psi/\partial y_2 \sim -AC(k)\,r_0^{-\frac{1}{2}}\cos(\tfrac{1}{2}\theta_0) \quad \text{as} \quad kr_0 \to 0. \tag{3.4}$$

Furthermore, since ψ has the same dimensions as the incident potential ϕ_i given by (3.1), it is clear that the constant $C(k)$ must take the form $C_0 k^{\frac{1}{2}}$, with C_0 depending only on the direction θ of the incident wave. Thus

$$\partial\psi/\partial y_1 \sim AC_0(\theta)\,(k/r_0)^{\frac{1}{2}}\sin(\tfrac{1}{2}\theta_0), \quad \partial\psi/\partial y_2 \sim -AC_0(\theta)\,(k/r_0)^{\frac{1}{2}}\cos(\tfrac{1}{2}\theta_0) \quad \text{as} \quad kr_0 \to 0. \tag{3.5}$$

It follows at once that a dipole of moment $d_j = (d_1, d_2)$, with incident field

$$\phi_i^{(\text{dipole})} = d_j\frac{\partial}{\partial y_j}\left\{\frac{\exp(ik|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|}\right\}, \tag{3.6}$$

induces a scattered potential $\psi(\mathbf{x})$ whose far field has the form

$$\psi^{(\text{dipole})} \sim C_0(\theta)\left(\frac{k}{r_0}\right)^{\frac{1}{2}}\frac{e^{ikr}}{r}\{d_1\sin(\tfrac{1}{2}\theta_0) - d_2\cos(\tfrac{1}{2}\theta_0)\} \quad (kr_0 \ll 1,\ kr \gg 1), \tag{3.7}$$

where $r = |\mathbf{x}|$ and $r_0 = |\mathbf{y}|$.

The field due to higher-order multipoles follows at once by simply differentiating with respect to $\mathbf{y}$, by means of the identities

$$\frac{\partial}{\partial y_1} = \cos\theta_0 \frac{\partial}{\partial r_0} - \frac{\sin\theta_0}{r_0}\frac{\partial}{\partial\theta_0}, \quad \frac{\partial}{\partial y_2} = \sin\theta_0 \frac{\partial}{\partial r_0} + \frac{\cos\theta_0}{r_0}\frac{\partial}{\partial\theta_0}.$$

Thus a quadrupole with components Q_{11}, Q_{12}, Q_{22} situated at $\mathbf{y}(r_0, \theta_0)$, with incident potential

$$\phi_i^{(\text{quad})} = Q_{jk}\frac{\partial^2}{\partial y_j \partial y_k}\left\{\frac{\exp(ik|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|}\right\}, \tag{3.8}$$

induces a scattered field ψ such that

$$\psi^{(\text{quad})} \sim C_0(\theta)\frac{k^{\frac{1}{2}}}{2r_0^{\frac{3}{2}}}\frac{e^{ikr}}{r}\{(Q_{22}-Q_{11})\sin(\tfrac{3}{2}\theta_0) + 2Q_{12}\cos(\tfrac{3}{2}\theta_0)\}, \tag{3.9}$$

and it is readily verified that the scattered fields of dipoles and quadrupoles with axes parallel to the scattering edge are negligible compared with (3.7) and (3.9). In particular, the expressions (3.8) and (3.9) show that the quadrupole source has a scattered field whose magnitude exceeds that of the incident field by the factor

$$|\psi^{(\text{quad})}/\phi_i^{(\text{quad})}| = O\{(kr_0)^{-\frac{3}{2}}\} \quad (kr_0 \ll 1,\ kr \gg 1), \tag{3.10}$$

and the enhancement of the sound measured by an observer at great distance is greater than the corresponding result obtained (2.11) for a finite scatterer.

The simple argument presented above gives the far field for points $\mathbf{x}$ that lie in the plane containing the source point $\mathbf{y}$ and perpendicular to the edge. For an observer at a general position $\mathbf{x}$, the solution can readily be generated from the special case (3.9) by replacing the plane wave (3.1) by a wave at oblique incidence, i.e.

$$\phi_i(\mathbf{y}) = A\exp\{-ik(y_1\cos\theta + y_2\sin\theta)\cos\varphi\},$$

where $A = e^{-iky_3\sin\varphi}\, e^{ik|\mathbf{x}|}/|\mathbf{x}|$, the essential difference being that the wavenumber k is replaced by its component in the (y_1, y_2) plane. Further, the result for a volume distribution of quadrupoles near the edge, having density q_{kj} per unit volume is obtained by integrating over the source region which is supposed finite. Thus

$$\psi \sim C_0(\theta)(k\cos\varphi)^{\frac{1}{2}}\frac{e^{ikr}}{2r}\int_V \{(q_{22}-q_{11})\sin(\tfrac{3}{2}\theta_0) + 2q_{12}\cos(\tfrac{3}{2}\theta_0)\}r_0^{-\frac{3}{2}}\,d\mathbf{y} \tag{3.11}$$

as $kr = k|\mathbf{x}| \to \infty$, evaluated over the source region V, and this gives the far field at a point (r, θ, ϕ) with direction cosines $(\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi)$ with respect to the source region. This result (3.11) has been obtained without any need to solve the full boundary-value problem for ψ; the r_0 and θ_0 dependence arises from the incompressible nature of the solution near the edge, and the important k dependence is given by simple dimensional analysis.

An exact solution is available for this particular problem, since the exact Green's function is known and can be expressed in terms of Fresnel integrals. Ffowcs Williams & Hall (1970) have exploited this result to show, in the present notation, that

$$\psi \sim -\frac{e^{-\frac{1}{4}i\pi}}{(2\pi)^{\frac{1}{2}}}\sin(\tfrac{1}{2}\theta)(k\cos\varphi)^{\frac{1}{2}}\frac{e^{ikr}}{2r}\int_V \{(q_{22}-q_{11})\sin(\tfrac{3}{2}\theta_0) + 2q_{12}\cos(\tfrac{3}{2}\theta_0)\}r_0^{-\frac{3}{2}}\,d\mathbf{y}, \tag{3.12}$$

which verifies (3.11) and provides further the θ dependence.

A result similar to (3.11) obtains readily for the *soft* half-plane on which the total potential is to be zero. In this case the far field has a similar form, with $C_0(\theta)$, $\sin(\frac{3}{2}\theta_0)$ and $\cos(\frac{3}{2}\theta_0)$ replaced by $D_0(\theta)$, $\cos(\frac{3}{2}\theta_0)$ and $-\sin(\frac{3}{2}\theta_0)$. The ratio of scattered against incident potential is again of the order $(kr_0)^{-\frac{3}{2}}$ for a quadrupole at distance r_0 from the edge, with axes perpendicular to the edge.

It should perhaps be mentioned here that a great deal of caution must be used in interpreting these model results, particularly those of the soft half-plane, as limiting cases for more general impedance type boundary conditions. It has been shown by Crighton & Leppington (1970) that if the half-plane is not quite perfectly soft, with the potential $\phi = \epsilon[\partial\phi/\partial y_2]$ proportional to the velocity discontinuity across the plate, then the limit $\epsilon \to 0$ is a singular one and does *not* yield the same result as the model problem with $\epsilon = 0$. The 'nearly hard' plate, on the other hand, with $\partial\phi/\partial y_2 = \epsilon[\phi]$, has a limit potential as $\epsilon \to 0$ that does tend uniformly to its model counterpart with $\partial\phi/\partial y_2 = 0$. Such problems with more general boundary conditions, thus involving coupled wave motions between wave-bearing surface and fluid, must be treated individually and will not be discussed here.

A further generalization of (3.11) is readily available for a wedge of exterior angle $p\pi/q$, for any rational number in the range $1 < p/q \leqslant 2$. The incompressible flow problem that determines the nature of the edge singularity is easily managed by means of a conformal transformation that maps the flow region on to a half space, and leads to the conclusion that the gradient of ψ is of the order $r_0^{-1+q/p}$, whence on dimensional grounds we have

$$|\nabla\psi| \sim AE_0(\theta)\,\frac{(kr_0)^{q/p}}{r_0} = \frac{e^{ikr}}{r}E_0(\theta)\,\frac{(kr_0)^{q/p}}{r_0}, \tag{3.13}$$

and the far field due to quadrupoles with axes perpendicular to the scattering edge is enhanced by the factor

$$|\psi^{(\text{quad})}/\phi_i^{(\text{quad})}| = O\{(kr_0)^{-2+q/p}\} \quad (kr_0 \ll 1,\, kr \gg 1), \tag{3.14}$$

in place of (3.10), for a wedge of exterior angle $p\pi/q$ with either a soft or hard boundary condition. Explicit details regarding the dependence on the source position (r_0, θ_0) can be derived along the lines of the analysis leading to (3.5) for the half-plate.

To interpret these results within the flow noise context, an order of magnitude argument of the kind discussed in §2 shows that the presence of such an edge modifies the 'U^8 law' of Lighthill to an extent even greater than that shown in §2 for finite bodies. The wave-number k is again replaced by U/lc, where U is a typical turbulence velocity and l a length scale of the turbulence, and the quadrupole strength Q_{kj}, proportional to the Reynolds stress $\rho u_k u_j$, is of order ρU^2. It follows that the scattered potential ψ for quadrupoles near a half-plate varies as $\psi \propto U^{\frac{5}{2}}$, whence the intensity I varies as

$$I \propto U^5 \tag{3.15}$$

for a soft or hard half-plate, compared with $I \propto U^8$ for unbounded flows, and $I \propto U^6$ for rigid bodies of size small compared with wavelength. For a wedge of more general exterior angle $p\pi/q$, $1 < p/q \leqslant 2$, it is found from (3.13) that

$$I \propto U^{4+2q/p} \quad (1 < p/q \leqslant 2), \tag{3.16}$$

and a more acute edge produces more intense scattering, as might be expected. In particular, for the half-plate, $p/q = 2$, and (3.15) is recovered; for a right-angled corner, $p/q = \frac{3}{2}$, whence

$$I \propto U^{16/3}. \tag{3.17}$$

The limit $p/q \to 1$ is evidently singular since (3.16) predicts a U^6 law, whilst a simple image argument for the infinite plane shows that U^8 is correct. This is not inconsistent, however, since if p/q is greater than unity, say $p/q = 1+\delta$ where δ is arbitrarily small, then the velocity at the edge is infinite for positive δ, but finite if $\delta = 0$.

In view of these results for scattering by wedges, it is natural to ask whether the large enhancement of sound is due essentially to the sharpness of the edge or to the great extent of the scattering surface. It is proposed here that, although the singularity inherent in the r_0 dependence is due to the edge geometry, the important wave-number dependence of (3.14) obtains only if the scatterer is large compared with wavelength. This conclusion is reached in the sections that follow by giving consideration, firstly, to the effect of finiteness of the scattering obstacle, and secondly, to the effect of smoothing out the sharpness of the edge.

4. Scattering by a circular disk

In order to examine the effect of finiteness, with regard to scattering by bodies having sharp edges, the rigid circular disk is chosen here as the simplest prototype problem of this type.

With source point $\mathbf{y}$ close to the edge and observation point $\mathbf{x}$ at great distance from the disturbance, the problem again reduces to that of finding the scattered potential $\psi(\mathbf{y})$ due to a plane wave, $\phi_i = A\exp(ik\boldsymbol{\alpha}.\mathbf{y})$, incident upon the disk, where $A = e^{ikr}/r$, $r = |\mathbf{x}|$ and $\boldsymbol{\alpha} = -\mathbf{x}/r$. Differentiation of ψ with respect to the point $\mathbf{y}$ then gives the far field at $\mathbf{x}$ due to higher-order sources at $\mathbf{y}$.

It has been shown in §3 how the incompressible nature of the flow in the immediate vicinity of an edge implies a singularity in the gradient of the potential, and this feature still holds with the present geometry, provided the point $\mathbf{y}$ is close to the edge compared with both wavelength and with radius a. That is, the derivatives of ψ take the general form

$$\frac{\partial \psi}{\partial y_1} \sim A\,\frac{C(k,a)}{r_0^{\frac{1}{2}}}\sin(\tfrac{1}{2}\theta_0);\; \frac{\partial \psi}{\partial y_2} \sim -A\,\frac{C(k,a)}{r_0^{\frac{1}{2}}}\cos(\tfrac{1}{2}\theta_0), \tag{4.1}$$

$kr_0 \ll 1$, $r_0/a \ll 1$, where $y_1 = r_0\cos\theta_0$ and $y_2 = r_0\sin\theta_0$ measure the source point $\mathbf{y}$ in the local co-ordinate system, of figure 1, based on the edge. Formula (4.1) is similar to (3.4), with the important difference that the unknown scaling constant C depends in the present case upon the radius a, in addition to its dependence upon wave-number k and direction of incidence.

It follows from (4.1), by direct differentiation with respect to $\mathbf{y}$, that a quadrupole source with incident potential

$$\phi_i^{(\text{quad})} = Q_{jk} \frac{\partial^2}{\partial y_j \partial y_k} \left\{ \frac{\exp(ik|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|} \right\} \tag{4.2}$$

induces a scattered potential $\psi(\mathbf{x}, \mathbf{y})$ with far field of the form

$$\psi^{(\text{quad})} \sim C(k,a) \frac{e^{ikr}}{r} \frac{1}{2r_0^{\frac{3}{2}}} \{(Q_{22}-Q_{11}) \sin(\tfrac{3}{2}\theta_0) + 2Q_{12} \cos(\tfrac{3}{2}\theta_0)\}, \tag{4.3}$$

which should be compared with the corresponding formula (3.9) for the rigid half-plane.

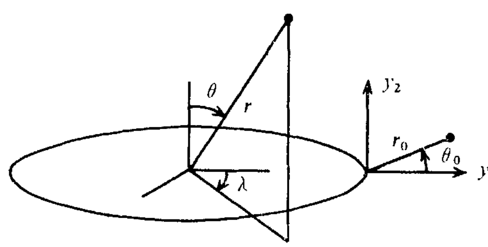

FIGURE 1. The polar co-ordinate system (r, θ, λ) and the local co-ordinates $y_1 = r_0 \cos \theta_0$, $y_2 = r_0 \sin \theta_0$, based on the edge.

The general form (4.3) has been dictated entirely by the incompressible nature of the flow near a sharp edge, and a similar result is readily obtained for the case of a soft disk. It remains to calculate the form of the multiplicative factor $C = C(k, a; \boldsymbol{\alpha})$, this constant having dimensions (length)$^{-\frac{1}{2}}$. Unlike the case of a semi-infinite plate, it is no longer possible to infer, in general, the functional dependence $C = C_0(\boldsymbol{\alpha})\, k^{\frac{1}{2}}$ on simple dimensional grounds, since for the disk there are *two* length scales, $1/k$ and a, involved in the parameter C. It is now argued that the form of C is essentially different according as the waves are short, or long, compared with the disk radius a.

For waves of length very small compared with radius, i.e. $ka \gg 1$, it is proposed that the field at a point $\mathbf{y}$, close to the edge, due to a plane incident wave depends to a first approximation only as the *local* geometry near $\mathbf{y}$. Thus the solution will be insensitive to the finite nature of the disk, which may therefore be replaced by the appropriate semi-infinite plane. In particular, the parameter $C(k, a)$ will be independent of a, and the far field (4.3) will be of order $k^{\frac{1}{2}}$.

To be more exact, define the spherical co-ordinate system (r, θ, λ) of figure 1 so that the disk is given by $r \leqslant a$, $\theta = \frac{1}{2}\pi$, $-\pi \leqslant \lambda \leqslant \pi$, with the source point $\mathbf{y}$ in the plane $\lambda = 0$ and close to the edge point $r = a$, $\theta = \frac{1}{2}\pi$, $\lambda = 0$. Then a quadrupole with incident potential (4.2) is found, by comparison with the half-plane result (3.12), to induce a far field of the form

$$\psi^{(\text{quad})}_{s} \sim \frac{e^{ikr}}{r} \frac{k^{\frac{1}{2}}}{r_0^{\frac{3}{2}}} G(\theta, \lambda) \exp\{-ika \sin\theta \cos\lambda - \tfrac{1}{4} i\pi\}$$
$$\times \{(Q_{22}-Q_{11}) \sin(\tfrac{3}{2}\theta_0) + 2Q_{12} \cos(\tfrac{3}{2}\theta_0)\}, \tag{4.4}$$

for $kr \gg 1$, $kr_0 \ll 1$, $ka \gg 1$, where G depends only on θ and λ.

The approximation is not valid at the 'grazing incidence' condition

$$\sin\theta\sin\lambda = 1;$$

this is to be expected, since the procedure of replacing disk by half-plane is invalid in this circumstance.

If the waves are much larger than the radius, on the other hand, i.e. $ka \ll 1$, it is clearly inappropriate to disregard the finite extent of the body, and in this limit the general theory of Curle, as outlined in the introductory section, seems the correct point of view. This procedure replaces the scatterer by an equivalent dipole layer whose density is proportional to the surface potential ϕ, this being calculated as though the flow were locally incompressible. In particular, since each dipole has far field proportional to k, then the integral of the layer over the finite disk has the same property, and the scattered field is proportional to wave-number. Explicit details regarding the strength of the far field can readily be calculated, if required, in terms of the solution of the limiting problem of a multipole near a rigid disk, in an incompressible fluid. Such static potential problems are classical, and can be solved by a variety of techniques documented by Sneddon (1966). Precise details of the solution are of secondary importance in the present analysis, compared with the crucial dependence upon wave-number k, and it suffices to merely quote the result that the scattered field induced by the quadrupole source (4.2) has the form

$$\psi^{(\text{quad})} \sim \frac{ik\cos\theta}{\pi}\frac{e^{ikr}}{r}\frac{a^{\frac{1}{2}}}{2r_0^{\frac{3}{2}}}\{(Q_{22}-Q_{11})\sin(\tfrac{3}{2}\theta_0)+2Q_{12}\cos(\tfrac{3}{2}\theta_0)\}, \tag{4.5}$$

for $kr \gg 1$, $kr_0 \ll ka \ll 1$, where θ is the polar angle of figure 1.

A comparison between the short-wave and long-wave limits (4.4) and (4.5) shows the main difference to be in their respective wave-number dependence, this being $O(k^{\frac{1}{2}})$ for short waves and $O(k)$ for long waves. For the case of a *soft* disk, the short wave limit has a far field $O(k^{\frac{1}{2}})$ while the long wave limit, obtained in terms of an equivalent monopole source layer over the disk, is $O(1)$. In the aerodynamic noise context, the usual identifications $Q_{ij} = O(\rho U^2)$, $k = O(U/lc)$, lead to the intensity laws $I \propto U^5$, in the short wave limit $ka \gg 1$, for hard or soft disks, with $I \propto U^6$ or $I \propto U^4$ in the long-wave limit $ka \ll 1$, according as the disk is hard or soft.

5. Effect of rounded edge

The formula (3.14) that pertains to the scattering by a wedge prompts the question as to how sharp must be the edge of the result to remain valid; any practical situation inevitably involves plates of finite thickness with geometries that are smoothed out at their edges. It is proposed here that the important k dependence of (3.14) remains unchanged if the edge is smoothed out over any distance that is small compared with wavelength.

To be definite, attention will be primarily confined to the case of a rigid semi-infinite plate of small but finite thickness, the model problem for this geometry being the semi-infinite plate of zero thickness described at length above. Similar remarks hold for any wedge of exterior angle $p\pi/q$, with either soft or hard boundary condition. It is supposed from the outset that the width d_1 of the plate and the maximum diameter d_2 of its smoothed end are very much less than a wavelength $2\pi/k$ (see figure 2), i.e. $kd \ll 1$, where d is the greater of d_1 and d_2.

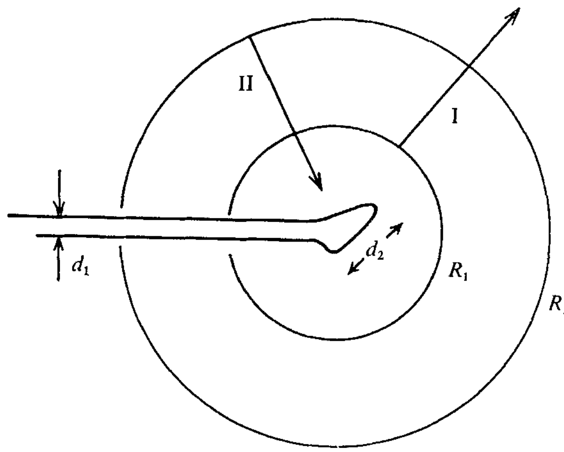

FIGURE 2. The length scale $d = \max(d_1, d_2)$ is small compared with wavelength $2\pi/k$. The radii R_1 and R_2 are such that $R_2 \gg d$ and $R_2 \ll 2\pi/k$.

Turning our attention to the reciprocal problem of finding the potential $\phi(\mathbf{y}) = \phi_i(\mathbf{y}) + \psi(\mathbf{y})$ due to an incident plane wave

$$\phi_i(\mathbf{y}) = A\exp\{-ik(y_1\cos\theta + y_2\sin\theta)\},$$

with $A = e^{ikr}/r$, it is convenient to divide the flow region into two overlapping domains (figure 2). In region I, at distances from the edge that are large compared with d, it is argued that the potential $\phi(\mathbf{y})$ fails to distinguish between the real geometry and its model counterpart, the half-plate of zero thickness. This half-plate solution $\phi_{\mathrm{I}}(\mathbf{y})$ is known explicitly (cf. Noble 1958), and in particular takes the asymptotic form, for points closer than a wavelength from the edge, given by

$$\phi_{\mathrm{I}}(\mathbf{y}) = 2A(kr_0)^{\frac{1}{2}}\pi^{-\frac{1}{2}}(1-\cos\theta)^{\frac{1}{2}}e^{-\frac{1}{4}i\pi}\sin(\tfrac{1}{2}\theta_0) + \text{constant} + o(kr_0)^{\frac{1}{2}} \quad \text{as} \quad kr_0 \to 0, \tag{5.1}$$

where $y_1 = r_0\cos\theta_0$ and $y_2 = r_0\sin\theta_0$. The constant is of no consequence since we are interested only in derivatives of ϕ with respect to $\mathbf{y}$.

In region II, consisting of points $kr_0 \ll 1$ that are close to the edge on a wavelength scale, the wave equation reduces to the Laplace equation, and we have to solve an incompressible flow problem, $\nabla^2\phi_{\mathrm{II}} = 0$, with zero normal derivative on the boundary, and a condition at infinity that is specified by matching the two approximations ϕ_{I} and ϕ_{II} in their common region $d \ll r_0 \ll 1/k$. It is seen from (5.1) that the required condition at infinity for the incompressible potential ϕ_{II} is that

$$\phi_{\mathrm{II}} \sim 2Ak^{\frac{1}{2}}\pi^{-\frac{1}{2}}(1-\cos\theta)^{\frac{1}{2}}e^{-\frac{1}{4}i\pi}r_0^{\frac{1}{2}}\sin(\tfrac{1}{2}\theta_0) \quad \text{as} \quad r_0/d \to \infty. \tag{5.2}$$

Precise details regarding ϕ_{II} clearly depend upon the exact geometry of the scatterer, but it is very important to observe that the parameter k appears only in the boundary condition (5.2) in the specifications for ϕ_{II}, and that the factor $2Ak^{\frac{1}{2}}\pi^{-\frac{1}{2}}(1-\cos\theta)^{\frac{1}{2}}e^{-\frac{1}{4}i\pi}$ appearing in (5.2) is merely a multiplicative constant that is maintained from the outer extremes of region II through to the scattering boundary itself. Thus the k dependence is fixed, and the shape of the body affects only the dependence of ϕ_{II} upon the geometrical variables r_0 and θ_0.

According to this argument, the scattered field $\psi(\mathbf{x})$ due to a multipole source at a point $\mathbf{y}$, closer than a wavelength to the edge, has a far field of the form

$$\psi(\mathbf{x}) \sim F(\mathbf{y})\,(e^{ikr}/r)\,k^{\frac{1}{2}}(1-\cos\theta)^{\frac{1}{2}} \quad \text{as} \quad kr = k|\mathbf{x}| \to \infty, \tag{5.3}$$

for $kd \ll 1$, $kr_0 \ll 1$, where the function F depends on the geometry of the end of the scatterer. The same form of solution holds also for the soft boundary condition. The scattered field (5.3) is seen to be similar to that of (3.7) and (3.9) for a thin half-plate, and in particular the wave-number dependence is such that the intensity law $I \propto U^5$ given by (3.15) is valid also for the present case of a thin plate with a rounded end, or indeed for an end of any shape, provided the thickness of the plate and the dimensions of its end section are small compared with wavelength. Similarly, the intensity law (3.16) holds for a wedge with smoothed end.

The arguments leading to these conclusions are mere conjectures, these being now supported by comparison with two situations for which exact results are available.

The problem of scattering a plane wave by a half-plate of finite thickness that occupies the region $y_1 < 0$, $-d < y_2 < 0$, has been solved by Jones (1953). This author, in particular, reduces the long wave problem ($kd \ll 1$) to that of an infinite system of linear equations for a set of numerical coefficients $A_1, A_3, A_5, \ldots$, that are related to the Fourier coefficients of the potential ϕ across the end face, $y_1 = 0$, $-d < y_2 < 0$.

According to the approach of the present work, on the other hand, a plane wave of incident potential

$$\phi_i(\mathbf{y}) = A\exp\{-ik(y_1\cos\theta + y_2\sin\theta)\}$$

induces a potential field that is indistinguishable from that of a semi-infinite plate of zero thickness, for points in region I ($r_0 = (y_1^2+y_2^2)^{\frac{1}{2}} \gg d$); in particular

$$\phi_{I} \sim 2A(kr_0)^{\frac{1}{2}}\,\pi^{-\frac{1}{2}}(1-\cos\theta)^{\frac{1}{2}}\,e^{-\frac{1}{4}i\pi}\sin(\tfrac{1}{2}\theta_0) + \text{constant},\ r_0 \ll 1/k. \tag{5.4}$$

In order to determine the approximate potential ϕ_{II} appropriate to region ($kr_0 \ll 1$), it remains to solve the simple incompressible flow problem

$$\nabla^2\phi_{II} = 0 \quad \text{in the flow,} \quad \partial\phi_{II}/\partial n = 0 \quad \text{on the surface,} \tag{5.5}$$

together with the condition at large distance that

$$\phi_{II} \sim A_0 r_0^{\frac{1}{2}}\sin(\tfrac{1}{2}\theta_0) \quad \text{as} \quad r_0 \to \infty, \tag{5.6}$$

to match with (5.4), wherein the constant $A_0 = 2Ak^{\frac{1}{2}}\pi^{-\frac{1}{2}}(1-\cos\theta)^{\frac{1}{2}}e^{-\frac{1}{4}i\pi}$ appears simply as a multiplicative constant throughout.

It is a routine matter to calculate the harmonic function ϕ_{II} satisfying (5.5) and (5.6) by means of conformal transformation and one can, in particular, readily derive the potential ϕ_{II} across the end face $y_1 = 0, \ -d < y_2 < 0$. A comparison with the results of Jones (1953) indicates that the potential ϕ_{II} so determined has the correct dependence on the variables k, d and θ; furthermore, a comparison of the numerical coefficients predicts that the constants $A_1, A_3, A_5, \ldots$, of Jones's work have the exact solution

$$A_{2n+1} = \frac{1}{2(2n+1)}\{J_n(n+\tfrac{1}{2}) - J_{n+1}(n+\tfrac{1}{2})\}, \tag{5.7}$$

in which J denotes a Bessel function. The first few values of (5.7), namely

$$A_1 = 0{\cdot}3481, \quad A_3 = 0{\cdot}0543, \quad A_5 = 0{\cdot}0229, \quad A_7 = 0{\cdot}0130,$$

agree convincingly well with the results,

$$A_1 \approx 0{\cdot}351, \quad A_3 \approx 0{\cdot}0559, \quad A_5 \approx 0{\cdot}0241, \quad A_7 \approx 0{\cdot}0140,$$

obtained by Jones by numerical computations.

A second problem that invites comparison with the present work is that of scattering by a pair of parallel thin half-plates given by $y_1 < 0, y_2 = 0$ and $y_1 < 0, y_2 = -d$, this being discussed at length by Noble (1958). For the long-wave limit ($kd \ll 1$), the outer approximation ϕ_{I} is the same as before and the near-field limit ϕ_{II} requires a solution of the Laplace equation, with zero normal derivative on each plate. Conditions at infinity are provided by the asymptotic requirements (5.6) as $r_0/d \to \infty$ outside the duct; a further condition $\phi_{\mathrm{II}} \sim 1$ as $r_0/d \to \infty$ inside the duct is obtained from the estimate

$$\phi \sim e^{-iky_1} \quad \text{as} \quad y_1/d \to -\infty, \tag{5.8}$$

as is readily established from the analysis given by Noble. Thus the form of the wave field near the edges is obtained in terms of this straightforward potential problem to be solved by conformal transformation. Details are unimportant, but it suffices to quote the particular result for the edge singularity, namely

$$\frac{\partial}{\partial y_2}\phi_{\mathrm{II}}(y_1, 0) \sim \frac{k^{\frac{1}{2}}(1-\cos\theta)^{\frac{1}{2}}\, e^{-\frac{1}{4}i\pi}}{2^{\frac{1}{2}}\pi^{\frac{1}{2}} y_1^{\frac{1}{2}}} \quad \text{as} \quad y_1 \to +0. \tag{5.9}$$

This is in complete agreement with the corresponding result implicit in Noble's work (when account is taken of an arithmetical error in the formula on p. 41 for $\Gamma(\alpha)/\Gamma(2\alpha)$ from which a factor $e^{\frac{1}{2}}$ should be omitted, this having bearing on the functions $K_{\pm}(\alpha)$, $L_{\pm}(\alpha)$ that appear in the analysis).

It is seen from (5.4) and (5.9) that the edge singularity for the duct has a coefficient $2^{-\frac{1}{2}}$ times that for a single plate; thus sources near an edge of the duct, at distances small compared with d, scatter slightly *less* sound than sources near the edge of a single plate.

Further, it is of some significance that the potential, according to (5.8), quickly assumes a plane wave form down the duct at distances that are large compared with d, but may be small compared with wavelength. By differentiating with respect to source point $\mathbf{y}$, it is clear that the scattered field from *any* quadrupole is at most $O(k^2)$, whether the axes are perpendicular or parallel to the edge. Evidently the region of intensely scattering sources, which is of order $r_0 \lesssim k^{-1}$ outside the duct, is reduced to $r_0 \lesssim d$ inside the duct. It is natural to anticipate that this property will hold also for the circular duct, and this result may have some bearing on the relative importance of turbulence sources inside and outside a jet exit.

Although detailed solutions have not been presented here, the methods of this section yield *explicit* results, in the long-wave asymptotic limit $kd \ll 1$, for any geometry such that the corresponding incompressible problem is amenable to solution by conformal transformation. The problem of scattering by a 'keyhole' obstacle, consisting of a wedge with a circular cylinder at its end, falls into this category, for example. It has been seen above, in connection with the half-plate of finite thickness, that the closed form solution obtained by this method has an advantage over the Wiener–Hopf approach, by which an infinite system of linear equations results.

On the other hand, an extension to axisymmetric geometries seems difficult, since the form of the outer flow field ϕ_{I} is not readily determined in an obvious way, and the powerful method of conformal transformation is no longer available to deal with the near field potential ϕ_{II}. Such a problem, concerning the semi-infinite circular rod, can be solved by means of the more general Wiener–Hopf technique, although the formidable nature of the preliminary manipulations and subsequent numerical computations are such that an alternative approach could well be worthwhile.

6. Conclusion

The distant field due to quadrupole sources is greatly increased by the presence of neighbouring scattering bodies. For obstacles that are small compared with wavelength, the scattered field is equivalent to an additional dipole, or source, according as the surface is hard or soft. In particular, for the prototype example of a sphere of radius a with a radial quadrupole at distance r_0 from the centre, the ratio of scattered against incident potential is given by

$$|\psi/\phi_i| = O(a^3/r_0^4 k), \quad \text{or} \quad |\psi/\phi_i| = O(a/r_0^3 k^2), \tag{6.1}$$

according as the surface is hard or soft. As expected, the enhancement is less if the sphere is situated further from the near field of the incident disturbance (i.e. as r_0 increases); if $r_1 - a \ll a$, then the sphere is in the near field of the incident quadrupole, and

$$|\psi/\phi_i| = O(1/ka), \quad \text{or} \quad |\psi/\phi_i| = O(1/k^2a^2). \tag{6.2}$$

To interpret this result within the flow noise context, one makes the usual identification $k \approx U/lc$ to express the wave-number k in terms of typical velocity U and length scale l associated with the turbulence, where c is the wave speed. Thus the potential is increased by the large factor $O(l/aM)$ for the hard sphere and $O(l^2/a^2M^2)$ for the soft sphere, where $M = U/c$ is a turbulence Mach number and is typically very small. Since the *intensity* of the distant sound field is proportional to the square of the potential, the U^8 law of Lighthill for unbounded flows must be modified to become a U^6 law or U^4 law for a hard or soft sphere. A similar analysis holds for any body of dimensions small compared with wavelength.

For quadrupoles close to, and with axes perpendicular to, the edge of a *wedge-like* body, whose dimension is large compared with wavelength and whose end may be smoothed out over a length scale small compared with wavelength, a completely different behaviour results. In such a case, the increase in far field potential takes the form

$$|\psi/\phi_i| = k^{-2+q/p}F(\mathbf{y}) \tag{6.3}$$

for either the hard or soft boundary condition, where the wedge-like body has exterior angle $(p/q)\,\pi, 1 < p/q \leqslant 2$. The parameter F has dimension $(\text{length})^{-2+q/p}$ and depends upon the source position $\mathbf{y}$ and upon details of the geometry. The explicit wave-number dependence given by formula (6.3) implies an increase in intensity by a factor proportional to $U^{-4+2q/p}$ and the eighth power law is replaced by

$$I \propto U^{4+2q/p}. \tag{6.4}$$

This applies, for example, to the case of a disk of radius a that is large compared with wavelength; further details are provided in the text for this case The analysis applies also to a large disk $(ka \gg l)$ of finite thickness d small compared with wavelength $(kd \ll l)$. It applies, for example, to the case of quadrupoles near the edge (but not near the corners) of a rectangular box if its dimensions are large scale and if its edges are smoothed out on a small scale, compared with wavelength. For this geometry $p/q = \frac{3}{2}$, whence $I \propto U^{\frac{16}{3}}$.

Finally, it is necessary to examine more closely the implications of infinite (or large) velocity induced at a sharp (or fairly sharp) edge. For a basic assumption in the Lighthill acoustic analogy is that the wave field induced by the turbulence has but little effect upon the Reynolds stress terms $\rho u_i u_j$ that are represented by incident quadrupole sources. Such an assumption is clearly invalid, without further justification, for geometrics with a sharp edge since the induced velocities are *infinite* there: indeed the precise nature of this singularity has played a key role in the analysis. The tendency towards high velocities near an edge will, of course, be controlled by the action of viscosity and it remains to show that the region in which viscosity is effective is much smaller than the near field of the edge, this being made up of distances from the edge on a scale much less than a wavelength.

A length scale l_ν to characterize the region affected by viscosity is obtained from the kinematic viscosity ν and frequency $\omega = kc$, whence

$$l_\nu = (\nu/\omega)^{\frac{1}{2}} \approx (\nu l/U)^{\frac{1}{2}}.$$

Thus the model presented herein is useful provided the viscous scale l_ν is very much less than a wavelength cl/U, i.e.

$$(U/c)\,(Ul/\nu)^{-\frac{1}{2}} \ll 1,$$

which is certainly the case when the Mach number U/c is small and the Reynolds number Ul/ν is large. Further, the model is then completely self-consistent.

For the 'eddy length' l represents the distance from the edge at which one can make the crude identifications $Q_{ij} \approx \rho U^2$, $\omega \approx U/l$, and consistency demands $l \gg l_\nu$, which is satisfied if $Ul/\nu \gg 1$. The viscous forces which, on a scale l_ν, ultimately reduce the velocity to zero on the body are therefore negligible on the scale l, which is the scale on which Q_{ij} has appreciable strength proportional to ρU^2.

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